

THE RATIONAL KHOVANOV HOMOLOGY OF 3-STRAND PRETZEL LINKS

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ABSTRACT. The 3-strand pretzel knots and links are a well-studied source of examples in knot theory. However, while there have been computations of the Khovanov homology of some sub-families of 3-strand pretzel knots, no general formula has been given for all of them. We give a general formula for the unreduced Khovanov homology of all 3-strand pretzel links, over the rational numbers.

1. INTRODUCTION

The goal of this paper is to compute the unreduced Khovanov homology, over \mathbb{Q} , of all 3-strand pretzel links. The literature contains computations for some pretzel knots; in [12], Suzuki computes the Khovanov homology of $(p, 2 - p, -r)$ pretzel knots with $p \geq 9$ odd and $r \geq 2$ even. More generally, the knots considered in [12] are quasi-alternating (see [1] and [4]), and for quasi-alternating links the Khovanov homology can be computed directly from the Jones polynomial and signature (see [8]). In [11], Starkston considers a family of non-quasi-alternating knots, the $(-p, p, q)$ pretzel knots for odd p and $q \geq p$. She computes the Khovanov homology of many of them. However, the Khovanov homology of most non-quasi-alternating pretzel knots has not appeared in the literature. We will complete this computation for all (non-quasi-alternating) pretzel links, over \mathbb{Q} .

As one may expect, we use the unoriented skein exact sequence in Khovanov homology; see [9] for a discussion of this sequence. We will use the sequence in an inductive argument, unraveling strands of a pretzel link one crossing at a time. In order to deal with all pretzel links, one must structure the induction carefully, as we will see. One must also use the Lee spectral sequence (see [6]) to help with many cases of the inductive step.

Before beginning, we will briefly review our conventions on 3-strand pretzel links. The (l, m, n) pretzel link will be denoted $P(l, m, n)$. (The standard letters to use are p , q , and r , but we have chosen the letters l , m , and n instead since q will be used for the quantum grading on Khovanov homology.) The knot $P(-3, 5, 7)$ is shown in Figure 1. It should remind the reader of the general form. For the link $P(l, m, n)$, positive values of l , m , and n represent right-handed strands, and negative values represent left-handed

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strands. Note that $P(l, m, n)$ is a knot when zero or one of $\{l, m, n\}$ are even, a two-component link when two of $\{l, m, n\}$ are even and one is odd, and a three-component link otherwise. The ordering of l , m , and n does not matter; it is clear that cyclic permutations of the three strands do not change the link, and transpositions simply amount to turning the link on its head.

For quasi-alternating links L , the Khovanov homology was computed to be thin in [8]. It depends only on the Jones polynomial and signature of L . If l , m , and n are all positive, then $P(l, m, n)$ is actually alternating, so its Khovanov homology is known (the Jones polynomials of pretzel links can be found in [5]). Also, mirroring the knot $P(l, m, n)$ gives $P(-l, -m, -n)$. Since Khovanov homology enjoys a symmetry under mirroring (the quantum and homological gradings are replaced by their negatives), we only need consider $P(-l, m, n)$ when l , m , and n are positive integers.

Champanerkar and Kofman [1] determined the quasi-alternating status of most pretzel links, and Greene [4] finished the rest. The following special case of Greene's results will be all we need:

Theorem 1.1 (Greene [4]). *$P(-l, m, n)$ is quasi-alternating if and only if $l > \min\{m, n\}$.*

Hence, to consider all non-quasi-alternating pretzel links, it will suffice to consider $P(-l, m, n)$ with $2 \leq l \leq m \leq n$.

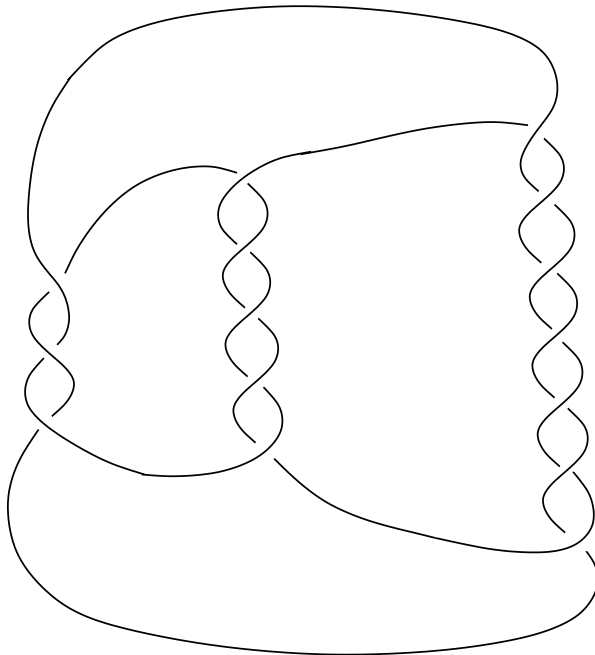


FIGURE 1. The $(-3, 5, 7)$ pretzel knot.

When we refer to Khovanov homology in this paper, we will always be using coefficients in \mathbb{Q} . The Khovanov homology of a link takes the form of a bigraded vector space over \mathbb{Q} . The two gradings will be denoted q , the quantum grading, and t , the homological grading. The bigraded vector space $Kh(P(-l, m, n))$ will be of the form $L \oplus U$, where L and U are spaces to be defined in Section 2. More precisely, we have

Theorem 1.2. *Suppose $2 \leq l \leq m \leq n$. Then*

$$Kh(P(-l, m, n)) = q^{\sigma_L} t^{\tau_L} L_{l,m,n} \oplus q^{\sigma_U} t^{\tau_U} U_{l,m,n},$$

where $L_{l,m,n}$ is a bigraded vector space specified in Definition 2.5, $U_{l,m,n}$ is a bigraded vector space specified in Definition 2.6, and the values of σ_L , τ_L , σ_U , and τ_U are specified in Proposition 2.8. (The multiplications by monomials in q and t simply shift the bigradings by the indicated amount).

The L summand will depend mostly on l and not on m or n , and the U summand will depend mostly on $m - l$ and $n - m$ (“mostly” means there are still some cases depending on parity and on whether $m = l$).

We will briefly sketch here how L and U are defined; the details are in Section 2. First, the spaces involved in the formula for $Kh(P(-l, m, n))$ will all be contained in three adjacent δ -gradings. In fact, all pretzel knots with arbitrarily many strands have the same property, as Champanerkar and Kofman point out in [1]: pretzel knots with arbitrarily many strands have Turaev genus 1 (there is a nice picture-proof in [1]), and unreduced homological width is bounded by the Turaev genus plus 2 (see [3], [2]).

Plot $Kh(P(-l, m, n)) = L \oplus U$ on a two-dimensional grid with the homological grading t on the x -axis and the quantum grading q on the y -axis (see Figure 2). The summands L and U will mostly occupy different regions, or regions which only slightly overlap, and L is to the left of U and below it. (Thus the q - and t -gradings of generators of L are generally lower than those of U , explaining the use of the letters L for “lower” and U for “upper.”) In the generic case $m \neq l$, the summands L and U share no columns of the grid when l is odd and two columns when l is even.

Each summand L and U will be contained in only two of the three possible δ -gradings. L will be contained in the higher two δ -gradings, and U will be contained in the lower two. Furthermore, L and U will each be made up of knight’s moves and exceptional pairs (see Figure 3). Hence, we only need to specify the values of L or U in one of its two δ -gradings, as long as we know where its exceptional pairs are. The required data are a sequence of integers, representing dimensions of the summand L or U in its bottom δ -grading, together with the bigrading of one generator of the summand (to fix the overall gradings) and the locations of the exceptional pairs. In Section 2, we will define L and U by specifying these data.

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2. THE GENERAL FORMULA

2.1. Bigraded vector spaces. Let V be a bigraded vector space. We will write $V = \oplus_{i,j} V_{i,j}$, where $V_{i,j}$ denotes the subspace in t -grading i and q -grading j . The space V can then be specified by its Poincaré polynomial, a Laurent polynomial P_V in q and t with positive integer coefficients such that the coefficient of $q^j t^i$ in P_V is the dimension of $V_{i,j}$ (note the ordering of the indices). We will henceforth identify V with P_V .

Monomial multiplication on P_V corresponds to grading-shift on V : if a and b are integers, then $q^a t^b V$ is defined as the bigraded vector space with Poincaré polynomial $q^a t^b P_V$. Equivalently,

$$(q^a t^b V)_{i,j} = V_{i-b, j-a}.$$

Visually, one depicts a bigraded vector space by drawing a grid (see Figure 2). Our convention will be to put the t -grading on the horizontal axis and the q -grading on the vertical axis (labelling only odd values or only even values of q , since the vector spaces in question will always have only odd or only even q -gradings).

The δ -grading on a bigraded vector space V is defined by $V_\delta = \oplus_{j-2i=\delta} V_{i,j}$ (see [9]). It corresponds to summing V along diagonals.

2.2. An example, and some definitions. Figure 2 depicts the Khovanov homology of $P(-3, 5, 7)$. It shows an important qualitative feature of the general formulas we will give: as discussed in the introduction, $Kh(P(-l, m, n))$ is made up of a “lower” summand and an “upper” summand with respect to the q - or t -grading. Out of the three allowable δ -gradings, the lower summand is contained in the highest two δ -gradings, and the upper summand is contained in the lowest two δ -gradings. For most pretzel links, the upper and lower summands overlap in at most two t -gradings (although the $(-l, l, n)$ pretzel links have a larger overlap when l is even).

We will now introduce a few definitions allowing us to efficiently package our formulas, following the discussion in the introduction. Each upper and lower summand is contained in two adjacent δ -gradings. Now, when a link has thin Khovanov homology (i.e. homology contained in two δ -gradings), the Lee spectral sequence from [6] tells us that the homology breaks into “knight’s moves” and “exceptional pairs” (see Figure 3). In our case, each of the summands L and U will individually break into knight’s moves and exceptional pairs. Thus we can specify the entire summand (up to overall grading) by, first, specifying what it is in the bottom δ -grading, and second, determining which generators in the bottom δ -grading are parts of exceptional pairs rather than knight’s moves.

Specifying the upper or lower summand in its bottom δ -grading amounts (up to shifts) to specifying a list of dimensions along the diagonal, or equivalently a sequence of integers. Since we will be shifting the grading later, we may as well work at first with spaces where the bottom δ -grading is $\delta = 0$. These observations motivate the following two definitions.

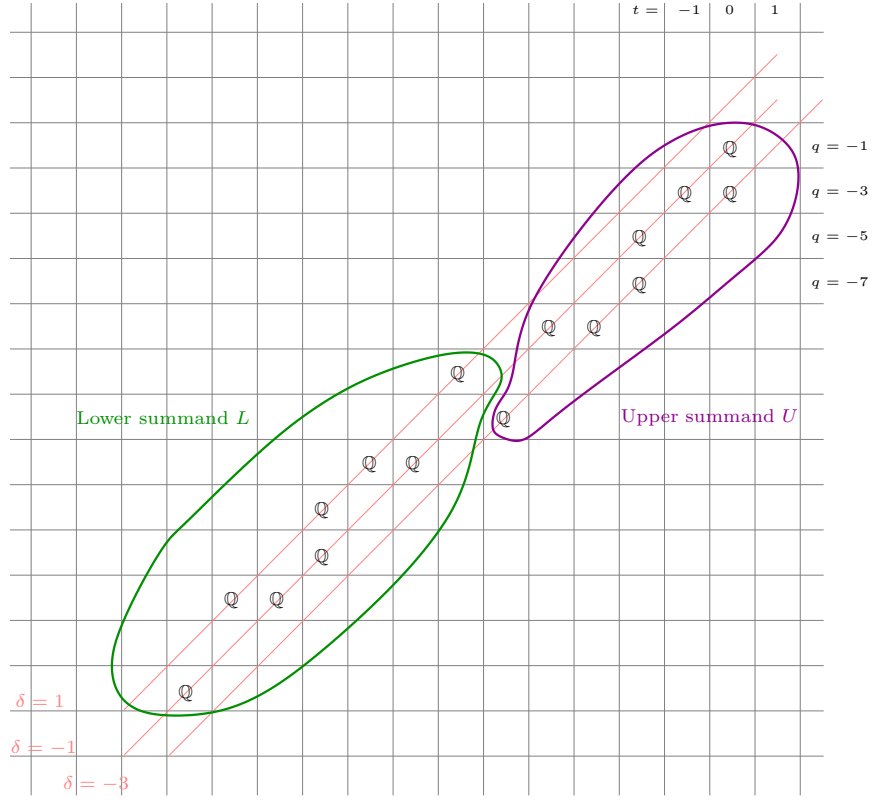
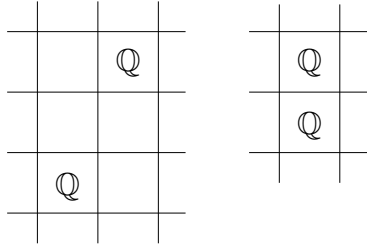
FIGURE 2. Rational Khovanov homology of the $(-3, 5, 7)$ pretzel knot.

FIGURE 3. A knight's move and an exceptional pair.

Definition 2.1. If $a = (\dots, a_{k-1}, a_k, a_{k+1}, \dots) : \mathbb{Z} \rightarrow \mathbb{N}$ is a sequence of positive integers with only finitely many nonzero entries, define the bigraded vector space $\tilde{V}[a]$ associated to a by

$$\tilde{V}[a] = \sum_{i=-\infty}^{\infty} a_{i+1} q^{4i} t^{2i}.$$

In other words, $\tilde{V}[a]$ is contained in δ -grading zero, with ranks $(\dots, a_{k-1}, a_k, a_{k+1}, \dots)$ along the diagonal. The term with rank a_1 has been placed at the origin. (We chose a_1 rather than a_0 to be the origin since the sequences used in most of our definitions will be supported on $\{1, 2, \dots, k\}$ for some k , and we would like the lowest nontrivial generator to have bigrading $(0, 0)$.)

The summands L and U will be grading-shifts of spaces $L_{l,m,n}$ and $U_{l,m,n}$; the spaces $L_{l,m,n}$ and $U_{l,m,n}$ will be contained in $\delta \in \{0, 2\}$ (recall that these δ -gradings are adjacent, since differences in q -gradings or δ -gradings are always even for a given link). To assemble these spaces, we need to know where their exceptional pairs are. Suppose we would like to specify V , a bigraded vector space made up of knight's moves and exceptional pairs and contained in $\delta \in \{0, 2\}$. Besides needing a sequence as in Definition 2.1, we also need some exceptional pair data. We will package the exceptional pair data of V in the form of a function $E : \mathbb{Z} \rightarrow \mathbb{N}$, where $E(i)$ is the number of exceptional pairs V has in t -grading i .

Definition 2.2. Let a be a finite sequence as in Definition 2.1 and let $E : \mathbb{Z} \rightarrow \mathbb{N}$ be any function such that $E(i) \leq a_i$. Define the sequence a' by $a'_i = a_i - E(i)$. Then

$$V[a, E] := (1 + q^4 t) \tilde{V}[a'] \oplus_{i=1}^k E(i) (1 + q^2) q^{4(i-1)} t^{2(i-1)}$$

In other words, we have singled out $E(i)$ generators in each index i and turned them into the bottom halves of exceptional pairs. Each other generator has been turned into the bottom half of a knight's move pair.

The definitions of the spaces $L_{l,m,n}$ and $U_{l,m,n}$ will involve sequences supported on $[1, k] := \{1, 2, \dots, k\}$ for some k . In these cases, we will specify $E : [1, k] \rightarrow \mathbb{N}$ by saying (for instance) “the exceptional pair is on the first index” or “there are two exceptional

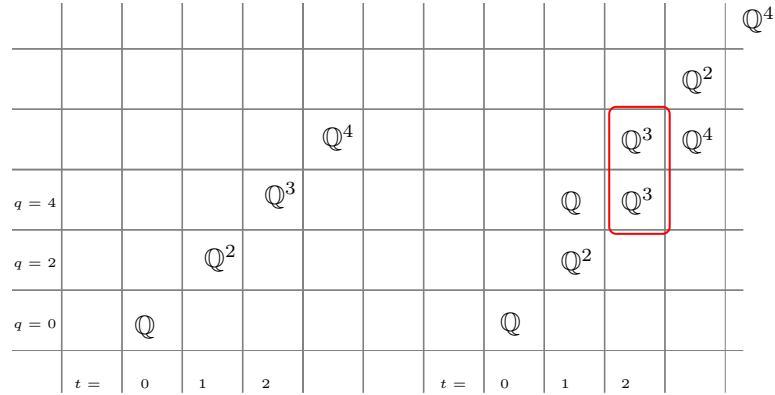


FIGURE 4. Let a be supported on $[1, 4]$, with values $(1, 2, 3, 4)$. On the left is $\tilde{V}[a]$, and on the right is $V[a, E]$, where there is one exceptional pair on the third index.

pairs, one on the first index and one on the second-to-last index.” If we wish to indicate that there are no exceptional pairs, we will simply omit E from the notation and just write

$$V[a] := (1 + q^4 t) \tilde{V}[a].$$

Definition 2.3. The following basic sequences will arise in our description of the upper and lower summands. All will be supported on $[1, k]$ for some k , and we will specify their values using k -tuples of positive integers.

- The sequence $a_k = (a_{1k}, \dots, a_{kk})$ of length k has the following pattern:
(1, 0, 2, 1, 3, 2, ...).
- The sequence $b_k = (b_{1k}, \dots, b_{kk})$ of length k has the following pattern:
(1, 0, 2, 0, 3, 1, 4, 2, 5, 3, ...).
- The sequence $c_k = (c_{1k}, \dots, c_{kk})$ of length k has the following pattern:
(1, 1, 2, 2, 3, 3, ...).

The sequences above are assumed to be truncated after the k^{th} entry, even if k is odd. If $k \leq 0$, the sequences are empty.

Definition 2.4. We will also use some operations on sequences supported on $[1, k]$ for various k . As before, such sequences will be specified with tuples of positive integers.

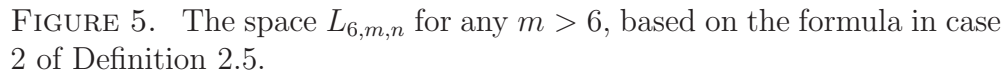
- If $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_l)$, then $a \cdot b$ denotes the concatenation $(a_1, \dots, a_k, b_1, \dots, b_l)$. Note that $a \cdot b$ is supported on $[1, k + l]$.
- If a is as above, a^m denotes $a \cdot a \cdot \dots \cdot a$, m times. For example, $(1)^m$ denotes the sequence $(1, \dots, 1)$ of length m .
- If a is as above, \bar{a} denotes a in reverse, i.e. (a_k, \dots, a_1) .
- If a and b are as above and $k = l$, then we can add a and b componentwise to get $a + b$. This makes sense even if b has negative coefficients, as long as the result has positive coefficients.

2.3. Formula for the lower summand. After these preliminaries, we can now define the space $L_{l,m,n}$ which will be the “lower” summand of $Kh(P(-l, m, n))$ after a grading shift. For $m \neq l$, there are two cases of the definition, depending only on the parity of l . There are four additional cases for $m = l$. The formulas are given below in the notation of the above subsection.

Definition 2.5. The lower summand $L_{l,m,n}$ is defined below in various cases. See Figure 5 for an example of case 2 below.

- (1) If $m \neq l$ and l is odd, then

$$L_{l,m,n} := V \left[a_{l-1} \cdot \left(\frac{l-1}{2} \right)^2 \cdot \overline{a_{l-1}} \right].$$


$$L_{l,m,n} := V \left[(1) \cdot c_{l-4} \cdot \left(\frac{l-2}{2} \right)^3 \cdot \left(\frac{l}{2} \right) \cdot \overline{b_l}, E \right],$$
$$L_{l,l,n} := V \left[a_{l-1} \cdot \left(\frac{l-1}{2} \right)^2 \cdot \overline{a_{l-1}} \cdot (0)^{n-l} \cdot (1), E \right],$$

where the exceptional pair is on the final index.

- (4) If $m = l$, l is even, and n is odd, then

$$L_{l,l,n} := V \left[(1) \cdot c_{l-4} \cdot \left(\frac{l-2}{2} \right)^3 \cdot \left(\frac{l}{2} \right) \cdot \overline{b_l} \cdot (0, 1)^{(n-l-1)/2}, E \right],$$

where the exceptional pair is on the first index.

- (5) If $m = l$, l is even, n is even, and $n \neq l$, then

$$L_{l,l,n} := V \left[(1) \cdot c_{l-4} \cdot \left(\frac{l-2}{2} \right)^3 \cdot \left(\frac{l}{2} \right) \cdot \overline{b_l} \cdot (0, 1)^{(n-l)/2}, E \right],$$

with exceptional pairs on the first and last indices.

- (6) If $m = l$, l is even, and $n = l$, then

$$L_{l,l,l} := V \left[(1) \cdot c_{l-4} \cdot \left(\frac{l-2}{2} \right)^3 \cdot \left(\frac{l}{2} \right) \cdot (\overline{b_l} + (\dots, 0, 0, 1)), E \right],$$

with one exceptional pair on the first index and two on the last index. The addition is done such that the last index of $\overline{b_l}$ gets the extra 1.

Note that to obtain the formulas when $m = l$, you just add some extra generators, in higher q - and t -gradings, to the formulas for $m \neq l$.

2.4. Formula for the upper summand. The rest of the Khovanov homology of $P(-l, m, n)$ comes from the “upper summand” $U_{l,m,n}$. It depends only on $g := m - l$ and $h := n - m$ as well as the parities of l , m , and n . Each of the eight choices for parities gives rise to a different formula, so the below definition has eight different cases.

Definition 2.6. Suppose $2 \leq l \leq m \leq n$ are integers; let $g = m - l$ and $h = n - m$.

- (1) If l , m , and n are odd,

$$U_{l,m,n} := V \left[a_g \cdot \left(\frac{g}{2} \right)^h \cdot \overline{a_g}, E \right],$$

where the one exceptional pair is on the final index.

- (2) If l and m are odd but n is even,

$$U_{l,m,n} := V \left[a_g \cdot \left(\frac{g}{2} \right)^{h-1} \cdot \overline{c_g}, E \right],$$

where the exceptional pair is on the first index of $\overline{c_g}$.

- (3) If l is odd, m is even, and n is odd,

$$U_{l,m,n} := V \left[a_{g-1} \cdot \left(\frac{g+1}{2}, \frac{g-1}{2} \right)^{(h+1)/2} \cdot \overline{c_{g-1}}, E \right],$$

where the exceptional pair is on the first instance of $\frac{g+1}{2}$.

- (4) If l is odd but m and n are both even,

$$U_{l,m,n} := V \left[a_{g-1} \cdot \left(\frac{g+1}{2}, \frac{g-1}{2} \right)^{h/2} \cdot \left(\frac{g+1}{2} \right) \cdot \overline{c_{g-1}}, E \right],$$

where the exceptional pairs are on the first and last instances of $\frac{g+1}{2}$. (If $h = 0$, this means there are two exceptional pairs in the same t -grading.)

- (5) If l is even but m and n are both odd,

$$U_{l,m,n} := V \left[b_{g+1} \cdot \left(\frac{g+1}{2}, \frac{g-1}{2} \right)^{h/2} \cdot \overline{c_{g-1}} \right].$$

- (6) If l is even, m is odd, and n is even,

$$U_{l,m,n} := V \left[b_{g+1} \cdot \left(\frac{g+1}{2}, \frac{g-1}{2} \right)^{(h-1)/2} \cdot \overline{c_g}, E \right],$$

where the exceptional pair is on the first index of $\overline{c_g}$.

- (7) If l and m are even but n is odd,

$$U_{l,m,n} := V \left[b_g \cdot \left(\frac{g+2}{2}, \frac{g-2}{2} \right)^{(h+1)/2} \cdot \overline{c_{g-1}}, E \right],$$

where the exceptional pair is on the first instance of $\frac{g+2}{2}$. If $l = m$, so $g = 0$, the -1 that arises here should be interpreted as zero.

- (8) If l , m and n are all even,

$$U_{l,m,n} := V \left[b_g \cdot \left(\frac{g+2}{2}, \frac{g-2}{2} \right)^{h/2} \cdot \left(\frac{g+2}{2} \right) \cdot \overline{a_g}, E \right],$$

where the exceptional pairs are on the first and last instances of $\frac{g+2}{2}$ and on the very last index. Again, if $g = 0$, the -1 in this formula should be interpreted as zero.

2.5. Orientations, grading shifts, and the general formula. Finally, we will put everything together with the appropriate grading shifts to produce the general formula for the Khovanov homology of pretzel links.

Before discussing grading shifts, though, we must decide on orientations, since changing the orientation of one component of a multi-component link changes the Khovanov homology by an overall grading shift.

The colored boxes in Figure 6 indicate the data needed to specify the orientation of $P(-l, m, n)$ in each case; for 3-component links like $P(-4, 6, 8)$, we can pick a direction in each box. For the pretzel links which are knots or 2-component links, not all choices of this data are allowable.

We will always orient the blue box upwards as shown in Figure 6. For knots, this choice fixes the entire orientation, according to Proposition 2.7 below. For links, we need

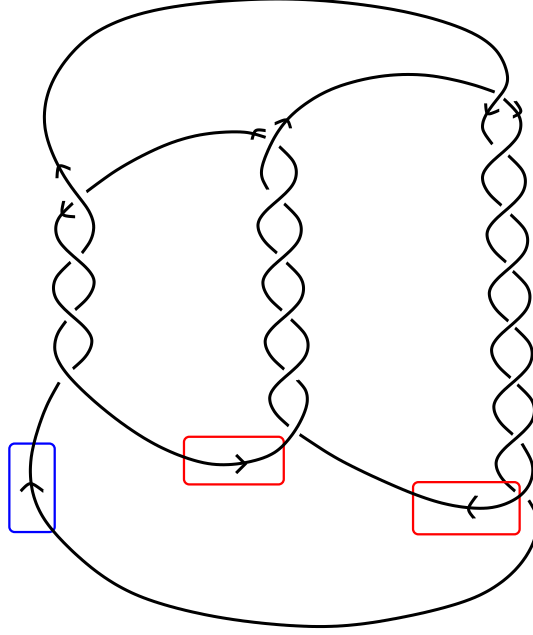


FIGURE 6. The link $P(-4, 6, 8)$, or $P(-4, 6, 8)_{RL}$, with the orientations we will use for even l . Note that all crossings are positive; this is a peculiarity of the RL orientation pattern.

to pin down the red boxes too. Each can point either right or left. We will indicate the way they point with subscripts. For example, when $P(-4, 6, 8)$ is oriented as in Figure 6, we will write it as $P(-4, 6, 8)_{RL}$.

When $P(-l, m, n)$ is a knot, the directions of the red boxes are fixed by our choice for the blue box:

Proposition 2.7. *Using the above notation:*

- If l , m , and n are odd, then $P(-l, m, n)$ is oriented RR .
- If l and m are odd but n is even, then $P(-l, m, n)$ is oriented LR .
- If l is odd, m is even, and n is odd, then $P(-l, m, n)$ is oriented LL .
- If l is even but m and n are odd, then $P(-l, m, n)$ is oriented RL .

When dealing with $P(-l, m, n)$ for even l , we will always orient the link in the RL manner as in Figure 6 (and omit the subscripts RL), in agreement with the orientation given by Proposition 2.7 when m and n are odd. This choice fixes orientations on all the links under consideration, except for $P(-l, m, n)$ when l is odd and m and n are both even. In this case, our inductive proofs will force us to consider both $P(-l, m, n)_{LL}$ and $P(-l, m, n)_{LR}$, and here we will be careful to indicate which one we mean.

Proposition 2.8. *The values of the grading shift variables σ_L , τ_L , σ_U , and τ_U in Theorem 1.2 depend only on the orientation pattern (RR, LL, RL, or LR) and are listed in Table 1.*

At this point we may restate our main theorem, having defined all of its components:

Theorem 2.9. *Suppose $2 \leq l \leq m \leq n$. Then*

$$Kh(P(-l, m, n)) = q^{\sigma_L} t^{\tau_L} L_{l,m,n} \oplus q^{\sigma_U} t^{\tau_U} U_{l,m,n},$$

where $L_{l,m,n}$ comes from Definition 2.5, $U_{l,m,n}$ comes from Definition 2.6, and the values of σ_L , τ_L , σ_U , and τ_U come from Proposition 2.8.

Definition 2.10. As above, we will sometimes write $Kh(P(-l, m, n)) = L \oplus U$. This means $L := q^{\sigma_L} t^{\tau_L} L_{l,m,n}$ and $U := q^{\sigma_U} t^{\tau_U} U_{l,m,n}$ as in Theorem 2.9. We will also say that L is “based at” (τ_L, σ_L) and U is “based at” (τ_U, σ_U) .

Corollary 2.11. *$Kh(P(-l, m, n))$ is contained in three δ -gradings which depend only on the orientation pattern (RR, LL, LR, or RL). These gradings are δ_{max} , $\delta_{max} - 2$, and $\delta_{max} - 4$, where δ_{max} is given in Table 1.*

2.6. An alternative approach to the grading shift data. The values in Table 1 may seem a bit mysterious. The remainder of this section will discuss an alternative way of specifying the values of σ_L , τ_L , σ_U , and τ_U in this table.

Write $Kh(P(-l, m, n)) = L \oplus U$ as in Definition 2.10. We could ask how much the t -gradings of L and U overlap. As it turns out, this difference follows a simple pattern. For the purpose of this section, we only care about the generic case $m \neq l$:

Proposition 2.12. *With the above notation, let Δ be the smallest t -grading in which U has a nonzero generator, minus the highest t -grading in which L has a nonzero generator. Suppose $m \neq l$. If l is odd, then $\Delta = 1$. If l is even, then $\Delta = -1$.*

We will take the values of δ_{max} in Table 1 as given. Since L is contained in the top two δ -gradings and U is contained in the bottom two δ -gradings, Proposition 2.12 pins

	σ_L	τ_L	σ_U	τ_U	δ_{max}	n_-
RR	$-2m - 2n - 1$	$-m - n$	$4l - 2m - 2n - 1$	$2l - m - n + 1$	1	$m + n$
LL	$-3l - 2m + n - 1$	$-l - m$	$l - 2m + n - 1$	$l - m + 1$	$n - l + 1$	$l + m$
LR	$-3l + m - 2n - 1$	$-l - n$	$l + m - 2n - 1$	$l - n + 1$	$m - l + 1$	$l + n$
RL	$n + m - 1$	0	$4l + m + n - 3$	$2l$	$n + m + 1$	0

TABLE 1. Values of the grading shift variables for the four different orientation possibilities. For convenience, the rightmost two columns also list the highest of the three δ -gradings with nonzero homology and the number n_- of negative crossings.

down how the L summand relates to the U summand. All that is left is to fix one overall reference point.

In other words, knowing the q - and t -gradings of any generator of the upper or lower summand will suffice to pin down the overall gradings of the Khovanov homology. Luckily, though, the t -gradings of exceptional pairs are easily computable through linking-number data:

Proposition 2.13 (Lee [7]). *If L is a knot, then $Kh(L)$ has an exceptional pair in $t = 0$. If L is a 2-component link with components L_1 and L_2 , then $Kh(L)$ has exceptional pairs in $t = 0$ and $t = 2lk(L_1, L_2)$. If L is a 3-component link with components L_1 , L_2 , and L_3 , then $Kh(L)$ has exceptional pairs in $t = 0$, $t = 2(lk(L_1, L_2) + lk(L_1, L_3))$, $t = 2(lk(L_1, L_2) + lk(L_2, L_3))$, and $t = 2(lk(L_1, L_3) + lk(L_2, L_3))$. These are all the exceptional pairs in $Kh(L)$.*

Once we know the t -grading of a generator of an exceptional pair, we can pin down its q -grading by knowing its δ -grading. But we know whether each exceptional pair is in L or in U , since this data was included in the definitions of L and U , and we know the δ -gradings of generators of L and U . Hence, given the values of δ_{max} in Table 1 and the overlap data in Proposition 2.12, we can populate the rest of Table 1 simply from the definitions of $L_{l,m,n}$ and $U_{l,m,n}$.

For example, consider the top row of Table 1. Looking at $Kh(P(-l, m, n))$ for odd values of $\{l, m, n\}$ with $m \neq l$ will be enough to fill in this row, since for these values $P(-l, m, n)$ is oriented RR . In this case, the summand L fills $2l + 1$ columns of the grid, because the sequence defining $L_{l,m,n}$ has length $2l$ and the knight's move pair on the far right end spills over into the next column. The summand U fills $-2l + m + n$ columns (the sequence defining $U_{l,m,n}$ in this case has length $2g + h = -2l + m + n$, and this time there is an exceptional pair on the far right).

Proposition 2.12 tells us that we should put the columns of U immediately to the right of the columns of L . The one exceptional pair is on the last column of U , so this column must be $t = 0$. Hence the first column of U is $t = 2l - m - n + 1$, and so $\tau_U = 2l - m - n + 1$. The one generator of U in this leftmost column is in $\delta = -3$, so its q -grading must be $4l - 2m - 2n - 1$. Hence $\sigma_U = 4l - 2m - 2n - 1$.

The summand L occupies $2l + 1$ columns to the left of $t = 2l - m - n + 1$, so the leftmost column of L is $t = -m - n$. The generator of L in this column lies in $\delta = -1$, so its q -grading is $-2m - 2n - 1$. Hence $\tau_L = -m - n$ and $\sigma_L = -2m - 2n - 1$. The rest of the table can be completed similarly.

3. PRELIMINARIES FOR THE PROOF

3.1. Skein sequences and cancellations. Our main computational tool will be the unoriented skein exact sequence in Khovanov homology, stated below.

Theorem 3.1. (See [9].) *Let D be a diagram for an oriented link, and consider a crossing c . One of the two resolutions of c , say D_o , is consistent with the orientations,*

and will be called the “oriented resolution.” One, say D_u , is not (the “unoriented” resolution). Let $\epsilon = n_-(D_u) - n_-(D)$, where n_- denotes the number of negative crossings in a diagram. Then, if c is a positive crossing, we have the sequence

$$\cdots \xrightarrow{f} q^{3\epsilon+2}t^{\epsilon+1}Kh(D_u) \longrightarrow Kh(D) \longrightarrow qKh(D_o) \xrightarrow{f} \cdots$$

If c is a negative crossing, we have the sequence

$$\cdots \xrightarrow{f} q^{-1}Kh(D_o) \longrightarrow Kh(D) \longrightarrow q^{3\epsilon+1}t^{\epsilon}Kh(D_u) \xrightarrow{f} \cdots$$

Schematically, the skein exact sequence will put us in the following situation: we have two known bigraded vector spaces V and W and a map $f : V \rightarrow W$ fitting in an exact sequence:

$$\cdots \xrightarrow{f} W \longrightarrow X \longrightarrow V \xrightarrow{f} W \longrightarrow \cdots,$$

where X is unknown. Our goal will be to determine X . We know that f preserves q -grading and increases t -grading by one. Thus, X arises from “cancelling” pairs of generators from $V \oplus W$ as in the following definition:

Definition 3.2. Let V and W be bigraded vector spaces. A cancellation of $V \oplus W$ is a subspace X of $V \oplus W$ obtained by eliminating “horizontal pairs,” i.e. two-dimensional subspaces $\mathbb{Q}v \oplus \mathbb{Q}w$ where $v \in V$, $w \in W$, v and w have the same q -grading, and the t -grading of w is one greater than the t -grading of v . For an example, see Figure 7.

Hence we can determine X by looking at all possible cancellations of $V \oplus W$ and rejecting all but one of them. We will accomplish this task by using the structure of Khovanov homology coming from the Lee spectral sequence (see [6]). For links with Khovanov homology contained entirely in three adjacent δ -gradings, this spectral sequence implies that the Khovanov homology breaks up into the knight’s moves and exceptional pairs discussed earlier (see Figure 3). Motivated by this fact, we make the following definition.

Definition 3.3. A bigraded vector space V is well-structured if it is a sum of knight’s moves and exceptional pairs.

For our sequences, all three spaces V , W , and X will be well-structured. Thus, in trying to determine X , we can first disregard all cancellations of $V \oplus W$ which are not well-structured. Out of the remaining options, it will turn out that we can uniquely determine the correct choice of X by looking at the number and placement of exceptional pairs. This general strategy will be implemented in the proofs below.

We can save ourselves some work with a general lemma, for which we need notation related to that of Section 2:

Definition 3.4. Let V be a well-structured vector space contained in two adjacent δ -gradings. Let $E_V : \mathbb{Z} \rightarrow \mathbb{N}$ be the function such that $E_V(i)$ is the number of exceptional

pairs of V in t -grading i , and let a_V be the sequence such that $V = V[a_V, E_V]$. Note that a_V is supported on the “actual” t -gradings of V , not necessarily on $[1, k]$ for any k .

Remark 3.5. Once E_V is well-defined, it is clear that a_V is well-defined, by subtracting values of E_V from ranks of V . However, it is a bit tricky to see why E_V is well-defined. One way to do this is as follows: look at the highest t -grading i of V . Note that V is contained in two δ gradings, so there are two possible q -gradings, say j and $j + 2$, corresponding to $t = i$. Then $E_V(k) = 0$ for $k > i$ and $E_V(i) = \dim V_{i,j}$. Now that we know $E_V(i)$, we can look at t -grading $i - 1$: $\dim V_{i-1,j-2} - E_V(i - 1) = \dim V_{i,j+2} - E_V(i)$, since both count the number of knight’s move pairs whose lower generator lies in bigrading $(i - 1, j - 2)$. Hence we can deduce the value of $E_V(i - 1)$. Continuing this process, E_V is well-defined. Furthermore, it is clear that $E_V(i)$ depends only on $V_{k,*}$ for $k \geq i$.

One could equivalently begin with the lowest t -grading rather than the highest one, and see that $E_V(i)$ alternatively depends only on $V_{k,*}$ for $k \leq i$.

Lemma 3.6. *In the above situation, suppose V and W are contained in the same two adjacent δ -gradings, and suppose also that $E_X \geq E_V + E_W$. Then in fact $X = V \oplus W$, i.e. no cancellations are possible.*

Proof. Suppose X is some nontrivial well-structured cancellation. Choose a cancelling pair of generators $(e \in V, f \in W)$ in the highest possible q -grading. Say the bigrading of f is (i, j) .

By well-structuredness of V and W , we have the following:

$$\begin{aligned} \dim V_{i,j} - E_V(i) &= \dim V_{i+1,j+4} - E_V(i+1); \\ \dim W_{i,j} - E_W(i) &= \dim W_{i+1,j+4} - E_W(i+1); \\ \dim X_{i,j} - E_X(i) &= \dim X_{i+1,j+4} - E_X(i+1). \end{aligned}$$

But since (e, f) cancels, we have

$$(1) \quad \dim X_{i,j} < \dim V_{i,j} + \dim W_{i,j}.$$

Since (e, f) is the highest pair to cancel, $\dim X_{i+1,j+4} = \dim V_{i+1,j+4} + \dim W_{i+1,j+4}$. By Remark 3.5, $E_X(i+1) = E_V(i+1) + E_W(i+1)$ since $X = V \oplus W$ in t -gradings $\geq i+1$. Hence, making the appropriate substitutions and cancellations in (1), we obtain

$$E_X(i) < E_V(i) + E_W(i),$$

contradicting our assumption. \square

In particular, if V and W have no exceptional pairs (i.e. E_V and E_W are identically zero), then there can be no nontrivial well-structured cancellation of $V \oplus W$.

One type of cancellation which occurs frequently is the following “standard cancellation:” suppose V and W are contained in the same two adjacent δ -gradings, with

$E_V(i) > 0$ and $E_W(i+1) > 0$. Then one can single out two generators each from V and W in the configuration of Figure 7. Cancelling the generators in the middle q -grading produces a knight's move, so the resulting space is still well-structured.

Remark 3.7. Below, we will often see a set of cancellations containing several standard cancellations, and we will want to show no more cancellations can occur. We can argue as follows: suppose we did some standard cancellations on $V \oplus W$ to get X , and then did another cancellation. The additional cancellation would, in fact, be a cancellation of $V' \oplus W'$, a proper subspace of X obtained from $V \oplus W$ by removing all four generators at each standard cancellation (rather than only those in the middle q -grading). The reason is that, after a standard cancellation, the two remaining generators under consideration are in the wrong δ -gradings to cancel (the generator of V is in the lower δ -grading, and the generator of W is in the higher one). Now, the spaces V' and W' are well-structured and have fewer exceptional pairs than V and W , so in many cases we will be able to apply Lemma 3.6 and derive a contradiction.

We can also identify a situation in which we can conclude a standard cancellation occurred, given some information about X :

Lemma 3.8. *Assume V and W are contained in the same two adjacent δ -gradings. Suppose i is the lowest t -grading of W and $E_W(i) > 0$. Assume also that $E_V(i-1) > E_X(i-1)$. Then a standard cancellation must have occurred between t -gradings $i-1$ and i .*

Proof. Let j be the lowest q -grading of W . As in the previous lemma, we have $\dim V_{i-2,j-4} - E_V(i-2) = \dim V_{i-1,j} - E_V(i-1)$ and $\dim X_{i-2,j-4} - E_X(i-2) = \dim X_{i-1,j} - E_X(i-1)$. We also know $V_{i-2,j-4} = X_{i-2,j-4}$ since i is the lowest t -grading of W .

Suppose no standard cancellation occurred; then $\dim X_{i-1,j} = \dim V_{i-1,j}$. After some substitutions, this equation becomes $E_X(i-1) - E_X(i-2) = E_V(i-1) - E_V(i-2)$. But since $X = V$ in t -gradings $\leq i-2$, we have $E_V(i-2) = E_X(i-2)$ by Remark 3.5. Thus $E_X(i-1) = E_V(i-1)$, a contradiction. \square

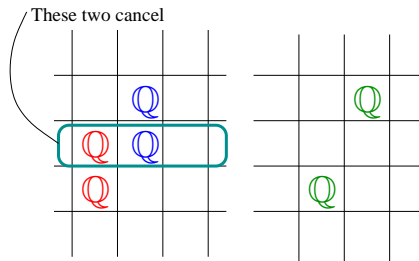


FIGURE 7. A standard cancellation. Red generators are from V , blue generators are from W , and green generators belong to X after the cancellation.

By interchanging V with W and replacing $i - 1$ with $i + 1$, we get the following:

Lemma 3.9. *Suppose i is the highest t -grading of V and $E_V(i) > 0$. Assume also that $E_W(i + 1) > E_X(i + 1)$. Then a standard cancellation must have occurred between t -gradings i and $i + 1$.*

3.2. Jones polynomial calculation. We need a Jones polynomial calculation to begin our inductive proofs. To fix notation, if L is a link, then $V_L(q^2)$ will denote the (normalized) Jones polynomial of L . (As usual, we will write everything in terms of the variable q , which squares to the standard argument of the Jones polynomial.) The unnormalized Jones polynomial $\bar{V}_L(q^2)$ is defined to be $(q + q^{-1})V_L(q^2)$.

Lemma 3.10. *Let $l \geq 2$. The unnormalized Jones polynomial of the link $P(-l, l, 0)_{LR}$ is*

$$\begin{aligned} \bar{V}_{P(-l, l, 0)_{LR}}(q^2) &= (-1)^l q^{2l+1} + \sum_{j=1}^{l-3} (-1)^{j+l+1} q^{2l-2j-1} + (2q + 2q^{-1}) \\ &\quad + \sum_{i=1}^{l-3} (-1)^{i+1} q^{-2i-3} + (-1)^l q^{-2l-1}. \end{aligned}$$

Remark 3.11. When l is odd, the orientation LR is forced by the choices we made earlier. When l is even, we will need to consider $P(-l, l, 0)_{RL}$, whose Jones polynomial differs from that of $P(-l, l, 0)_{LR}$ by an overall factor of q . To pin down this factor, note that $P(-l, l, 0)_{RL}$ is (after flipping the diagram over) just $P(-l, l, 0)_{LR}$ with the orientation of the “outer” component reversed. The linking number of this component with the rest of the link in $P(-l, l, 0)_{LR}$ is $-l/2$. Hence $P(-l, l, 0)_{RL}$ picks up a factor of $q^{-6(l/2)} = q^{3l}$. The resulting formula for the unnormalized Jones polynomial of the link $P(-l, l, 0)_{RL}$ is

$$\begin{aligned} (2) \quad \bar{V}_{P(-l, l, 0)_{RL}}(q^2) &= (-1)^l q^{5l+1} + \sum_{j=1}^{l-3} (-1)^{j+l+1} q^{5l-2j-1} + (2q^{3l+1} + 2q^{3l-1}) \\ &\quad + \sum_{i=1}^{l-3} (-1)^{i+1} q^{3l-2i-3} + (-1)^l q^{l-1}. \end{aligned}$$

Proof of Lemma 3.10. The link $P(-l, l, 0)_{LR}$ is a connected sum of the positive right-handed $(2, l)$ torus link $T_{2,l}$ and its mirror $\overline{T_{2,l}}$, the left-handed negatively oriented $(2, l)$ torus link. Since the Jones polynomial is multiplicative under connected sum, we can obtain the Jones polynomial of $P(-l, l, 0)_{LR}$ easily. We start with the well-known

formula $V_{T(2,l)}(q^2) = q^{l-1} + \sum_{i=1}^{l-1} (-1)^{i+1} q^{l+2i+1}$. Therefore

$$\begin{aligned}
V_{P(-l,l,0)_{LR}}(q^2) &= \left(q^{-l+1} + \sum_{i=1}^{l-1} (-1)^{i+1} q^{-l-2i-1} \right) \left(q^{l-1} + \sum_{j=1}^{l-1} (-1)^{j+1} q^{l+2j+1} \right) \\
&= 1 + \sum_{i=1}^{l-1} (-1)^{i+1} q^{-2i-2} + \sum_{j=1}^{l-1} (-1)^{j+1} q^{2j+2} + \sum_{i=1}^{l-1} \sum_{j=1}^{l-1} (-1)^{i+j} q^{2j-2i} \\
&= (-1)^l q^{2l} + \sum_{j=1}^{l-2} (-1)^{j+l} j q^{2l-2j} - (l-2)q^2 + l - (l-2)q^{-2} \\
&\quad + \sum_{i=1}^{l-2} (-1)^{i+1} (l-i-1) q^{-2i-2} + (-1)^l q^{-2l}.
\end{aligned}$$

Multiplying by $(q + q^{-1})$, we get the above formula for the unnormalized Jones polynomial. \square

3.3. Proof strategy. We will now outline how the proof of the general formula will be structured. Consider a crossing in the standard diagram for $P(-l, m, n)$. One resolution of the crossing produces another pretzel link, with either l , m , or n reduced by one. The other resolution produces a torus link whose Khovanov homology is known. Hence, given an appropriate base case for induction, the skein exact sequence of the crossing relates two known entities (the Khovanov homology of a torus link and of a smaller pretzel link) with the unknown entity we would like to compute.

Thus, the most naive idea for a proof might be to pick one strand of $P(-l, m, n)$ and unravel it, one crossing at a time, until we reach a quasi-alternating link whose Khovanov homology we know. One could hope that at each step, the skein exact sequence provides enough data to reduce the computation for the larger pretzel link to the computation for the smaller one, inductively determining $Kh(P(-l, m, n))$.

Unfortunately, the sequence does not always contain enough data; there are some ambiguities. For example, suppose we tried to unravel the middle strand to reach the quasi-alternating link $P(-l, l-1, n)$. An ambiguity would arise in trying to determine $P(-l, l, n)$ from $P(-l, l-1, n)$. If l or n is even, a further ambiguity arises in trying to determine $P(-l, l+1, n)$ from $P(-l, l, n)$.

One way around these ambiguities is to unravel the middle strand as far as possible, and then unravel the rightmost strand until reaching a quasi-alternating link. For odd l and n , this amounts to a series of reductions from $P(-l, m, n)$, to $P(-l, l, n)$, to $P(-l, l, l-1)$. Everything in this procedure works, as we will see below.

For even l or even n , this strategy would mean going from $P(-l, m, n)$, to $P(-l, l+1, n)$, to $P(-l, l+1, l-1)$. If l is even, each step works out. However, if l is odd and n is even, another ambiguity arises: the skein sequence does not uniquely determine $Kh(P(-l, l+1, l+2))$ from $Kh(P(-l, l+1, l+1))$. Luckily, though, by this time we

already know $P(-l, l+1, n)$ for all odd n . Hence we only need to go from $P(-l, l+1, n)$ to $P(-l, l+1, n-1)$ when n is even. In this case the skein exact sequence does give us enough data.

We will organize the proof as follows: first, we will consider the case when l is odd. We will prove the formula for $P(-l, l, n)$ and then complete the proof for the case of odd l and n . We will next deduce the formula for $P(-l, l+1, n)$ (even n) from the formula for $P(-l, l+1, n-1)$. Then we will derive the general formula for $P(-l, m, n)$ with odd l .

For even l , the roadmap is a bit simpler. We will prove the formulas for $P(-l, l, n)$ and $P(-l, l+1, n)$ first, and then deduce the general formula for $P(-l, m, n)$.

4. PROOF OF THE GENERAL FORMULA FOR ODD l

4.1. $P(-l, l, n)$ for odd l . We begin with the special case $m = l$. A glance at Definition 2.6 reveals that $U_{l,l,n} = 0$ for odd l . Corollary 2.11 tells us we should be proving that the Khovanov homology lies in $\delta = 1$ and $\delta = -1$, with the form specified in Definition 2.5. More precisely, we have the following lemma, which holds for all n (not just $n \geq l$):

Lemma 4.1. *Let $l \geq 3$ be odd and let $n \geq 0$. Let $a^{(n)}$ denote the sequence $a_{l-1} \cdot (\frac{l-1}{2})^2 \cdot \overline{a_{l-1}}$ plus an extra 1 in the $(l+n)^{\text{th}}$ spot. (When $n \geq l$ this is the sequence used in the definition of $L_{l,l,n}$.) Then*

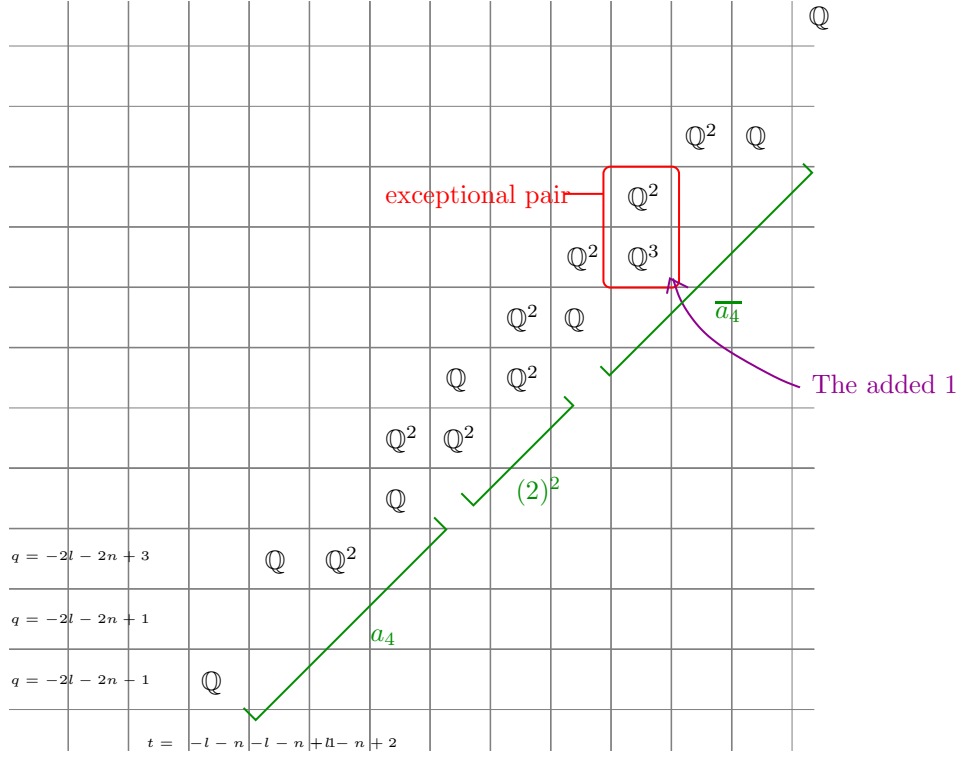
$$Kh(P(-l, l, n)) = q^{-2l-2n-1} t^{-l-n} V[a^{(n)}, E],$$

where the exceptional pair is in the $(l+n)^{\text{th}}$ index (where the extra 1 was added). See Figure 8 for an example (the case $l = 5$ and $n = 2$).

Note that this formula is consistent with our general formula, regardless of whether n is even or odd.

Proof. We will induct on n , starting with $n = 0$. By Lemma 2.3 of [1], $P(-l, l, 0) = \overline{T_{2,l}} \# T_{2,l}$ is quasi-alternating since $T_{2,l}$ is alternating. Hence, its Khovanov homology is contained in two δ -gradings. These must be $\delta = \pm 1$ since $P(-l, l, 0)$ is slice; in fact, $P(-l, l, n)$ is slice for all n (see [11] for a nice explanation with pictures).

Now, by the thinness just established, the Khovanov polynomial of $P(-l, l, 0)$ is uniquely determined by the requirement that it be well-structured and consistent with its Jones polynomial. To check that $q^{-2l-1} t^{-l} V[a^{(0)}, E]$ is consistent with the Jones polynomial formula given in Lemma 3.10, plug in $t = -1$. The knight's moves (i.e. the

FIGURE 8. The Khovanov homology of $P(-5, 5, 2)$.

sequence *without* the added 1) give us

$$\begin{aligned}
 -q^{-2l-1}(1-q^4) & \left(\sum_{i=0}^{\frac{l-3}{2}} \left((i+1)q^{4i} - iq^{4i+2} \right) + \left(\frac{l-1}{2} \right) (q^{2l-2} - q^{2l}) \right. \\
 & \left. + \sum_{i=0}^{\frac{l-3}{2}} \left(\left(\frac{l-3}{2} - i \right) q^{2l+4i+2} - \left(\frac{l-1}{2} - i \right) q^{2l+4i+4} \right) \right)
 \end{aligned}$$

which expands to

$$\begin{aligned}
 & -q^{-2l-1} \left(1 + \sum_{i=0}^{l-4} (-1)^i q^{2i+4} - q^{2l} - q^{2l+2} + \sum_{i=0}^{l-4} (-1)^{i+1} q^{2l+2i+6} + q^{4l+2} \right) \\
 & = -q^{-2l-1} + \sum_{i=1}^{l-4} (-1)^{i+1} q^{-2l+2i+4} + (q^{-1} + q) + \sum_{i=0}^{l-4} (-1)^i q^{2i+5} - q^{2l+1}.
 \end{aligned}$$

The exceptional pair adds $q^{-1} + q$, and after reindexing this is precisely the formula given by Lemma 3.10.

Now suppose the lemma holds for $P(-l, l, n-1)$, and consider $P(-l, l, n)$. We will use the skein exact sequence obtained by resolving the top crossing on the rightmost strand. This crossing is negative, so we must use the second sequence in Theorem 3.1. The standard diagram we use for $P(-l, l, n)$ has $l+n$ negative crossings, as listed in Table 1. Note that $P(-l, l, n)$ is oriented RR if n is odd and LR if n is even, but the corresponding values of n_- in Table 1 turn out to be the same since $l = m$. Similarly, the unoriented resolution, $P(-l, l, n-1)$, has $l+n-1$ negative crossings. Thus, $\epsilon = -1$. The oriented resolution is a diagram for the 2-component unlink U_2 , which has Khovanov polynomial $q^2 + 2q + q^{-2}$. The sequence is

$$\xrightarrow{f} q^{-1}Kh(U_2) \longrightarrow Kh(P(-l, l, n)) \longrightarrow q^{-2}t^{-1}Kh(P(-l, l, n-1))f \xrightarrow{f} .$$

For convenience, call the left-hand term W , the middle term X , and the right-hand term V . W has two exceptional pairs in $t = 0$, and V has one exceptional pair in $t = -1$. X has one exceptional pair in $t = 0$.

By induction, plus the grading shifts in the sequence, $V = q^{-2l-2n-1}t^{-l-n}V[a^{(n-1)}, E]$. The map f preserves the q -grading and increases the t -grading by one.

Note that $V \oplus W$ looks like the answer we want for X , except that it has three exceptional pairs rather than one. Two of them come from W and have $t = 0$, while the third comes from the exceptional pair in V and has $t = -1$. In fact, as discussed above, X is a cancellation of $V \oplus W$, and the cancellation will cut us down to one exceptional pair. Let $\{e_1, e_2, e_3, e_4\}$ be any basis for W , in q -gradings 1, -1 , -1 , and -3 respectively (see Figure 9). We must determine which of the e_i cancel with a generator of V . If we could show that e_1 and e_2 survive while e_3 and e_4 cancel, this would imply the formula we want.

First, e_1 cannot cancel because $V_{1,-1} = 0$ (there is nothing “to the left of e_1 ” in V). Next, note that $\dim V_{-1,-1} = \dim V_{-5,-2} + 1$ because of the exceptional pair. But we must have $\dim X_{-1,-1} = \dim X_{-5,-2}$ since the only exceptional pair of X lies in $t = 0$. So a one-dimensional subspace of $\mathbb{Q}(e_2, e_3)$ must cancel (or, since we never pinned down e_2 and e_3 , we can just say that e_3 cancels).

Finally, if e_4 did not cancel, then it would need to be in an exceptional pair in X , since we know $\dim X_{1,1} = \dim V_{1,1} = \dim V_{5,2} = \dim X_{5,2}$. But since the q -grading of e_4 is -3 , this would imply that the s -invariant of $P(-l, l, n)$ is -2 . This contradicts the sliceness of $P(-l, l, n)$ mentioned above, since Rasmussen proves in [10] that slice knots must have $s = 0$. Hence e_4 must cancel, and we have proved our formula for $Kh(P(-l, l, n))$. \square

Remark 4.2. Alternatively, the results of Greene in [4] imply that $P(-l, l, n)$ is quasi-alternating for $n < l$, so for these values of n we could obtain the above result simply by looking at the Jones polynomial. However, the general formula for the Jones polynomial of pretzel knots (computed by Landvøy in [5]) is a bit complicated, and beginning the induction in Lemma 4.1 at $n = 0$ rather than $n = l - 1$ makes for a cleaner argument.

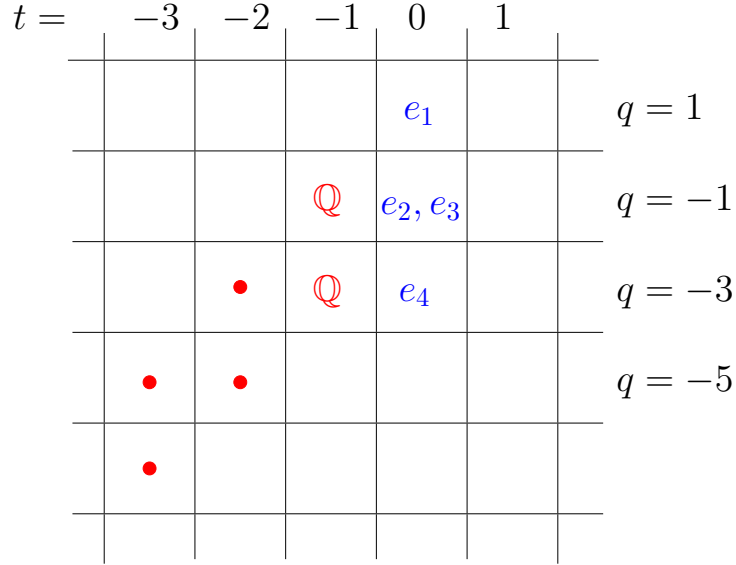


FIGURE 9. The inductive step of Lemma 4.1. The red copies of \mathbb{Q} depict the exceptional pair of V . The red dots are meant to suggest that V has generators to the left of the two copies of \mathbb{Q} ; note that it may also have generators to the right of these copies of \mathbb{Q} as well, if n is small.

4.2. $P(-l, m, n)$ for odd l and odd n .

Theorem 4.3. *The formulas given in Section 2 hold for $P(-l, m, n)$ when l and n are odd.*

Proof. We will use the base case $m = l$ to induct. Assume our formula holds for $P(-l, m - 1, n)$. We will prove it for $P(-l, m, n)$ using the skein exact sequence for the top crossing in the middle strand, which is a negative crossing regardless of whether m is even or odd.

First assume m is even. Our diagram for $P(-l, m, n)$ is oriented LL and so has $l + m$ negative crossings (see Table 1). The unoriented resolution is $P(-l, m - 1, n)$, and its diagram is oriented RR with $m + n - 1$ negative crossings. Hence $\epsilon = n - l - 1$. The oriented resolution is a diagram for the (right-handed, positively oriented) torus link $T_{n-l, 2}$. The sequence is

$$\xrightarrow{f} q^{-1}Kh(T_{n-l, 2}) \longrightarrow Kh(P(-l, m, n)) \longrightarrow q^{3n-3l-2}t^{n-l-1}Kh(P(-l, m-1, n)) \xrightarrow{f}.$$

Again, call the left-hand term W , the middle term X , and the right-hand term V . W has exceptional pairs in $t = 0$ and $t = n - l$. V has an exceptional pair in $t = n - l - 1$. X has one exceptional pair in $t = 0$.

Since V is the Khovanov homology of a pretzel knot (up to a shift), we may write $V = L \oplus U$ as in Definition 2.10. By induction, and because of the grading shifts in the above sequence, L is based at $(q, t) = ((-m - (n - 1)) + (n - l - 1), (-2m - 2(n - 1) - 1) + (3n - 3l - 2)) = (-l - m, -3l - 2m + n - 1)$. Similarly, U is based at $(l - m + 1, l - 2m + n - 1)$. Note that these values match up with the LL row of Table 1, which is good since $P(-l, m, n)$ is oriented LL .

To analyze the cancellations, we will fix some notation. As we just noted, W has two exceptional pairs, one in $t = 0$ and the other in $t = n - l$. Pick generators $\{e_1, e_2\}$ for the first exceptional pair and $\{f_1, f_2\}$ for the second, such that e_1 and f_1 have the higher q -gradings (Figure 10 shows the case $m = l + 1$).

First consider the case $m = l + 1$, as in Figure 10. No generators of W except for the e_i and f_i could possibly cancel. If either of the e_i 's cancelled, X could not have an exceptional pair in $t = 0$, a contradiction. On the other hand, both f_i 's must cancel for X to be well-structured, since X has no exceptional pairs except in $t = 0$.

The result of these cancellations is that, first, L loses its exceptional pair. Definition 2.5(1) shows that this behavior is precisely what was predicted for the lower summand of $P(l, l + 1, n)$, in contrast with Definition 2.5(3). We already saw that L was based at the correct point. Figure 10 shows that we also pick up an upper summand from W' , where W' denotes W without its top exceptional pair. Recall that V did not start with an upper summand. The new upper summand is based at $(0, n - l - 3)$, in accord with row LL of Table 1. It has the correct form as specified in Definition 2.6(3), since the formula there in the case $g = 0$ just gives the Khovanov homology of $T_{n-l,2}$ (minus the top exceptional pair), up to a grading shift.

Now consider the case $m > l + 1$ (m is still assumed to be even). L is based at $t = -l - m$, and it occupies $2l + 1$ columns, so the highest t -grading of L is $l - m$. But this value is less than -1 , while the lowest t -grading in W is 0 . Hence no cancellations between L and W can occur, and we have $X = L \oplus X'$ where X' is a cancellation of $U \oplus W$. Note that both U and W are contained in the same two adjacent δ -gradings, namely $\delta = n - l - 1$ and $\delta = n - l - 3$.

Now, $Kh(P(-l, m - 1, n))$ has zero as its highest t -grading (its one exceptional pair lies in its highest t -grading, so this grading must be 0). Hence V (or equivalently U) has $n - l - 1$ as its highest t -grading, and it has an exceptional pair in this column. Also, W has one of its two exceptional pairs in $t = n - l$. Lemma 3.9 applies, and there is a standard cancellation between $t = n - l - 1$ and $t = n - l$.

Now Remark 3.7 applies, and any further cancellations would be cancellations of $U' \oplus W'$ (using the notation of Remark 3.7). But U' no longer has any exceptional pairs, and W' only has one of them. The exceptional pair of X must be contained in its upper summand X' , so it must come from $U' \oplus W'$. Thus, by Lemma 3.6, no further cancellations are possible.

It is easy now to verify our formula inductively. Recall that U began its life in the form of Definition 2.6(1). The standard cancellation just replaces the exceptional pair

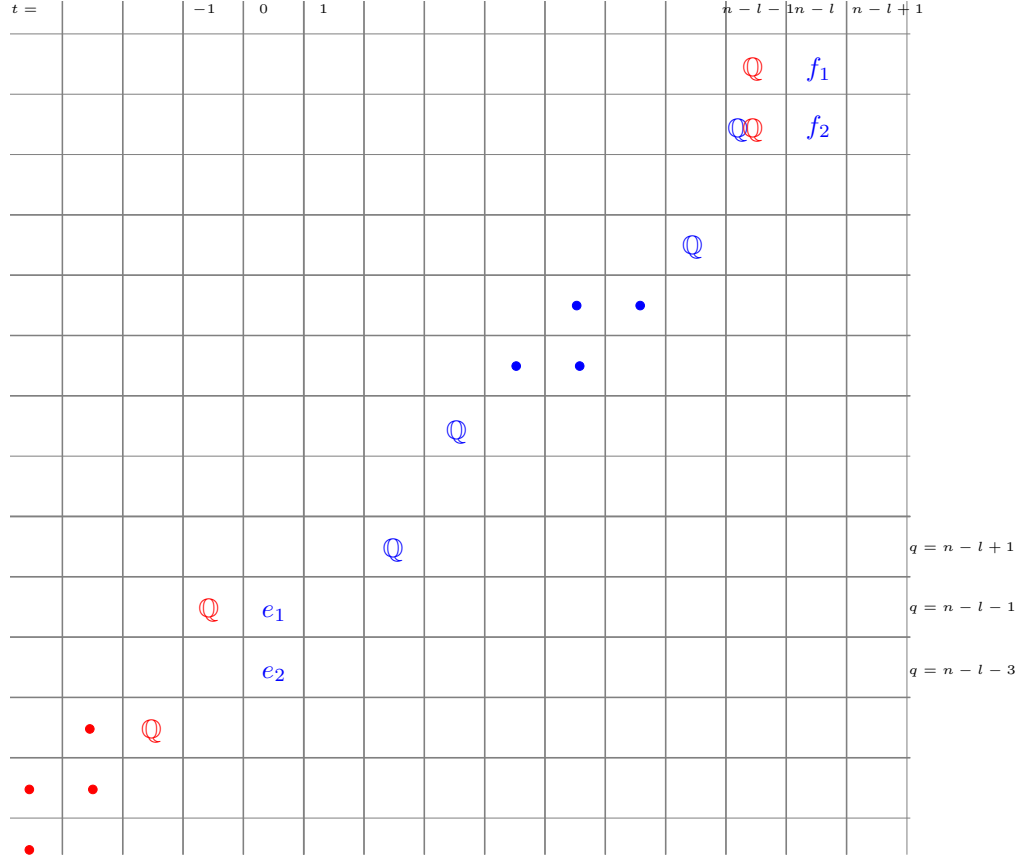


FIGURE 10. The case $m = l + 1$ in Theorem 4.3. Red denotes V and blue denotes W .

of U with a knight's move; in effect, U becomes $V[a]$ (up to a shift), where a is the sequence $a_{m-1-l} \cdot (\frac{m-1-l}{2})^{n-m+1} \cdot \overline{a_{m-1-l}}$. Adding in the rest of W amounts to adding the sequence $(1, 0)^{(n-l)/2}$, coordinate-wise, to the right side of a , where the first 1 carries an exceptional pair. ("To the right side" means the addition is done such that the final 0 of $(1, 0)^{(n-l)/2}$ lines up with the last nonzero index of a .) This addition replaces $\overline{a_{m-l-1}}$ with $\overline{c_{m-l-1}}$ and $(\frac{m-1-l}{2})^{n-m+1}$ with $(\frac{m-l+1}{2}, \frac{m-l-1}{2})^{(n-m+1)/2}$. The exceptional pair is on the first instance of $\frac{m-l+1}{2}$, giving us the upper summand we want as specified in Definition 2.6(3). We noted before that the summands are based at the right points, so we have proved our formula.

Now assume m is odd. The diagram for $P(-l, m, n)$ is oriented RR and has $m + n$ negative crossings, while the diagram for the unoriented resolution $P(-l, m - 1, n)$ is oriented LL and has $l + m - 1$ negative crossings. Thus $\epsilon = l - n - 1$. The oriented resolution is $-T_{n-l, 2}$, the negatively oriented right-handed $(n - l, 2)$ torus link. The

skein sequence is

$$\xrightarrow{f} q^{-1}Kh(-T_{n-l,2}) \longrightarrow Kh(P(-l, m, n)) \longrightarrow q^{3l-3n-2}t^{l-n-1}Kh(P(-l, m-1, n)) \xrightarrow{f}.$$

Again, call the left-hand term W , the middle term X , and the right-hand term V . W has exceptional pairs in $t = l - n$ and $t = 0$, while V has an exceptional pair in $t = l - n - 1$. X has an exceptional pair in $t = 0$. Write $V = L \oplus U$. By induction, and the grading shifts in the sequence, L is based at $(-m - n, -2m - 2n - 1)$ and U is based at $(2l - m - n + 1, 4l - 2m - 2n - 1)$. These values agree with the RR row of Table 1.

The analysis of the cancellations proceeds as above. The lowest t -grading in W is $t = l - n$, where it has an exceptional pair, and V has its exceptional pair in $t = l - n - 1$. Lemma 3.8 ensures a standard cancellation occurs between $t = l - n$ and $t = l - n - 1$. Afterwards, Remark 3.7 and Lemma 3.6 apply, and no more cancellations can occur (in particular, no cancellation occurs in the highest t -grading). Easy checks now verify our formulas, as before. \square

4.3. $P(-l, l + 1, n)$ for odd l and general n . Now we will compute $Kh(P(-l, l + 1, n))_{LL}$ for even n , given that we know our formula holds for odd n .

Lemma 4.4. *For $n \geq l + 1$, $Kh(P(-l, l + 1, n))_{LL}$ is given by the formula in Theorem 2.9.*

Proof. We have already proved the result for odd $n > l + 1$ in Theorem 4.3. Hence, we may assume n is even. Resolve the top crossing on the rightmost strand, a positive crossing. Our diagram for $P(-l, l + 1, n)_{LL}$ has $2l + 1$ negative crossings, while the unoriented resolution (a diagram for the unknot) has $l + n - 1$ negative crossings. Hence $\epsilon = n - l - 2$. The oriented resolution is a diagram for $P(-l, l + 1, n - 1)$. The skein sequence is

$$\xrightarrow{f} q^{3n-3l-4}t^{n-l-1}Kh(U) \longrightarrow Kh(P(-l, l + 1, n)) \longrightarrow qKh(P(-l, l + 1, n - 1)) \xrightarrow{f},$$

where U is the unknot, with Khovanov polynomial $q + q^{-1}$. Denote the left, middle, and right terms by W , X , and V respectively. V has an exceptional pair in $t = 0$ and W has an exceptional pair in $t = n - l - 1$. X has exceptional pairs in both $t = 0$ and $t = n - l - 1$ by Proposition 2.13, since the linking number of the two components of $P(-l, l + 1, n)_{LL}$ is $\frac{n-l-1}{2}$.

Suppose $n > l + 1$, and write $V = L \oplus U$. It is clear that L and U are based at the right points to form the lower and upper summands for $Kh(P(-l, l + 1, n))_{LL}$, by looking at Table 1 and the grading shift of q in the above sequence. The situation is shown in Figure 11. There is no cancellation since $V_{*, n-l-1} = 0$, $\dim W_{*, n-l-1} = 2$, and X has an exceptional pair in $t = n - l - 1$.

If $n = l + 1$, then V comes from the Khovanov homology of $P(-l, l + 1, l)$, which was not covered in Theorem 4.3. However, $P(-l, l + 1, l) = P(-l, l, l + 1)$, and we computed

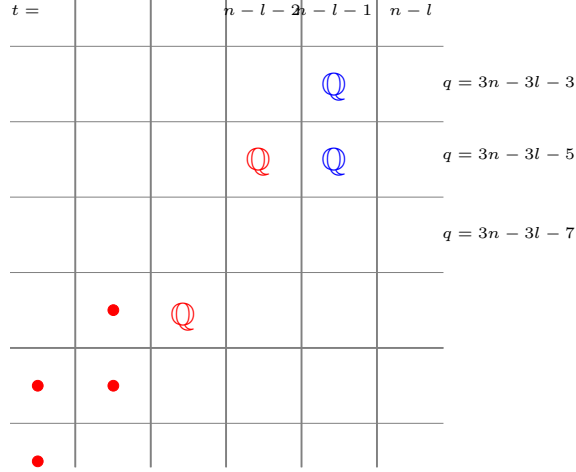


FIGURE 11. The case $n > l + 1$ in Lemma 4.4. Red denotes V and blue denotes W .

the Khovanov homology of this knot in Lemma 4.1. So we still know what V is, by looking at the formula above for $P(-l, l, l + 1)$.

In fact, there is still no cancellation in the sequence. We have $\dim V_{*,0} = \dim W_{*,0} = 2$, but X needs two exceptional pairs in $t = 0$. So X is just $V \oplus W$. One can check that this result agrees with our general formula. \square

4.4. $P(-l, m, n)$ for odd l and even n .

Theorem 4.5. *When l is odd and n is even, $Kh(P(-l, m, n)_{LR})$ is given by the formula in Theorem 2.9.*

Proof. Our base case is $m = l + 1$, where we can appeal to Lemma 4.4. Note that for $P(-l, m, n)$ with l odd, m even, and n even, switching orientations from LL to LR picks up a factor of $q^{-6((n-m)/2)}t^{-2((n-m)/2)} = q^{3m-3n}t^{m-n}$, since the linking number of the relevant component with the rest of the link is $(n-m)/2$. Comparing this shift with the data in Table 1, we see that our formulas hold for $P(-l, m, n)_{LR}$ as soon as they hold for $P(-l, m, n)_{LL}$. In particular, our formulas hold for $P(-l, l + 1, n)_{LR}$.

Now assume $m > l + 1$. The diagram for $P(-l, m, n)_{LR}$ has $l + n$ negative crossings. We will resolve the top crossing on the middle strand, a positive crossing. The unoriented resolution, a diagram for the right-handed torus knot $T_{n-l,2}$, has $l + m - 1$ negative crossings. Thus $\epsilon = m - n - 1$. The oriented resolution is a diagram for $P(-l, m - 1, n)_{LR}$, and the skein sequence is

$$\xrightarrow{f} q^{3m-3n-1}t^{m-n}Kh(\overline{T_{n-l,2}}) \longrightarrow Kh(P(-l, m, n)) \longrightarrow qKh(P(-l, m - 1, n)) \xrightarrow{f}.$$

Call the left-hand term W , the middle term X , and the right-hand term V . W has one exceptional pair in $t = m - n$. If m is even, V has one exceptional pair in $t = 0$ and X has exceptional pairs in $t = 0$ and $t = m - n$. Similarly, if m is odd, V has exceptional pairs in $t = 0$ and $t = m - n - 1$, and X has one exceptional pair in $t = 0$.

Write $V = L \oplus U$. It is easy to see, by looking at the LR row of Table 1 and at the shift of q in the above sequence, that L and U are based at the correct points. The exceptional pair, or pairs, of V fall in the U summand.

We need only consider cancellations of $U \oplus W$. The exceptional pair of W is in $t = m - n$. When m is odd, U has two exceptional pairs ($t = 0$ and $t = m - n - 1$). When m is even, U has one exceptional pair in $t = 0$. Hence, when m is even, no cancellations can occur by Lemma 3.6 (note that we only need to consider the case $m \leq n$), and it is easy to check our formula. When m is odd, Lemma 3.8 guarantees a standard cancellation between $t = m - n - 1$ and $t = m - n$, while Remark 3.7 and Lemma 3.6 preclude any further cancellations. Again, one can now check our formula, finishing the computation. \square

5. PROOF OF THE GENERAL FORMULA FOR EVEN l

5.1. $P(-l, l, n)$ for even l . We now carry out the strategy of Section 3.3 for even l . We first compute $Kh(P(-l, l, n))$; recall that we are using the RL orientation throughout Section 5. This computation will not be used as a base case for induction, but we need to deal with it anyway since it is not covered by our other computations.

Theorem 5.1. *Let $l \geq 2$ be even and let $n \geq 0$. Let c denote the sequence $(1) \cdot c_{l-4} \cdot (\frac{l-2}{2})^6 \cdot \overline{c_{l-4}} \cdot (0, 1)$. For even $n \geq 0$, let $d^{(n)}$ denote the sequence $(1, -1, \dots, 1, -1, 2)$ of length $n + 1$ (when $n = 0$, $d^{(0)}$ is just the sequence (2)). For odd $n \geq 1$, let $d^{(n)}$ be the sequence $(1, -1, \dots, 1)$ of length n . Define*

$$c^{(n)} = c + d^{(n)},$$

where the addition to c starts in the $(l + 1)^{st}$ spot.

Then if $n < l$,

$$Kh(P(-l, l, n)_{RL}) = q^{l+n-1} V[c^{(n)}, E],$$

where there are exceptional pairs in the first index and the last index, plus two exceptional pairs in the $(l + n + 1)^{st}$ index if n is even.

If $n \geq l$, then $Kh(P(-l, l, n)_{RL})$ is as described in Theorem 2.9.

Proof. The proof is by induction on n , as before. When $n = 0$, we know $P(-l, l, 0)_{RL}$ is quasi-alternating, and its Jones polynomial was given in Equation (2). A check similar to that in Lemma 4.1 verifies the formula.

Suppose the lemma holds for $n - 1$, and consider $P(-l, l, n)$. Again, we will use the skein sequence from resolving the top crossing on the rightmost strand, a positive crossing. Our diagram for $P(-l, l, n)$ has no negative crossings. The unoriented resolution, a diagram for the 2-component unlink, has $l + n - 1$ negative crossings (regardless of

the orientation chosen). Hence $\epsilon = l + n - 1$. The oriented resolution is a diagram for $P(-l, l, n - 1)_{RL}$. The skein sequence for a positive crossing is

$$\xrightarrow{f} q^{3l+3n-1}t^{l+n}Kh(U_2) \longrightarrow Kh(P(-l, l, n)) \longrightarrow qKh(P(-l, l, n - 1)) \xrightarrow{f} .$$

Denote the left, middle, and right terms by W , X , and V respectively. W has two exceptional pairs in $t = l + n$. If n is odd, V has four exceptional pairs (one in $t = 0$, one in $t = 2l$, and two in $t = l + n - 1$), and X has two exceptional pairs (one in $t = 0$ and one in $t = 2l$). On the other hand, if n is even, then V has two exceptional pairs (in $t = 0$ and $t = 2l$), and X has four exceptional pairs (one in $t = 0$, one in $t = 2l$, and two in $t = l + n$). The map f preserves the q -grading and increases the t -grading by one.

First, suppose n is odd and $n < l$; see the left side of Figure 12 for reference. Note that $d^{(n)}$ is just $d^{(n-1)}$ with the terminating 2 replaced by a 1, and thus $c^{(n)}$ is related to $c^{(n-1)}$ in a similar way.

Hence $V \oplus W$ looks like the answer we want for X , except that it has six exceptional pairs rather than two. In fact, the cancellation process will cut us down to two exceptional pairs. Let $\{e_1, e_2, e_3, e_4\}$ be any basis for W , in q -gradings $3l + 3n + 1$, $3l + 3n - 1$, $3l + 3n - 1$, and $3l + 3n - 3$ respectively. We must determine which of the e_i cancel with a generator of V . If we could show that e_1 survives while the rest cancel, this would imply the formula we want (by a simple check).

First, e_1 cannot cancel because there is nothing “to the left of e_1 ” in V . Next, note that $\dim V_{l+n-1, 3l+3n-1} = \dim V_{l+n-2, 3l+3n-5} + 2$ because of the exceptional pairs in V .

$t =$	$l + n - 2$	$l + n - 1$	$l + n$		$t =$	$2l - 1$	$2l$	$2l + 1$		$t =$	$l + n - 2$	$l + n - 1$	$l + n$	
			e_1					e_1					e_1	$q = 3l + 3n + 1$
		\mathbb{Q}^2	e_2, e_3				\mathbb{Q}^2	e_2, e_3				\mathbb{Q}	e_2, e_3	$q = 3l + 3n - 1$
	\bullet	\mathbb{Q}^2	e_4			\bullet	\mathbb{Q}^3	e_4			\bullet	\mathbb{Q}^2	e_4	$q = 3l + 3n - 3$
\bullet	\bullet				\bullet	\bullet	\mathbb{Q}			\bullet	\bullet	\mathbb{Q}		
\bullet					\bullet					\bullet	\bullet			
										\bullet				

FIGURE 12. The case of odd n in Theorem 5.1. The left side depicts the case $n < l$. The middle depicts $n = l + 1$. The right side depicts the case $n > l + 1$. As usual, red denotes V and blue denotes W .

But we must have $\dim X_{l+n-1, 3l+3n-1} = \dim X_{l+n-2, 3l+3n-5}$ since X has no exceptional pair in $t = l + n - 1$ or $t = l + n - 2$. So both e_2 and e_3 must cancel.

Finally, if e_4 did not cancel, then we would have $\dim X_{l+n-1, 3l+3n-3} = \dim X_{l+n, 3l+3n+1} + 1$. But we need $\dim X_{l+n-1, 3l+3n-3} = \dim X_{l+n, 3l+3n+1}$, again because the only exceptional pairs of X are in $t = 0$ and $t = 2l$. Hence e_4 must cancel, and we have proved our formula when n is odd and less than l .

The next case for odd n is $n = l + 1$. The middle of Figure 12 is a reference here. A similar argument implies that e_2 , e_3 , and e_4 all cancel, and a quick check verifies that our results for the lower and upper summands of X agree with Definition 2.5(4) and Definition 2.6(7).

If n is odd and $n > l + 1$, then the right side of Figure 12 depicts the situation. The generator e_1 still cannot cancel. But logic very similar to before implies that e_4 and a one-dimensional subspace of (e_2, e_3) must cancel, since X has no exceptional pairs in $t = l + n$. Thus, the lower summand loses an exceptional pair (as predicted by Definition 2.5), and the upper summand turns an exceptional pair into a knight's move.

Finally, suppose n is even; luckily, this case is easier. First, if $n < l$, we want to show that only e_4 cancels (using the above notation). Again, e_1 cannot cancel. If some combination of e_2 and e_3 cancelled, then we would have $\dim X_{l+n-1, 3l+3n-1} < \dim X_{l+n-2, 3l+3n-5}$, an impossibility since X has no exceptional pairs in $t = l + n - 2$. So both e_2 and e_3 survive.

Since $n < l$, $P(-l, l, n)$ is quasi-alternating by [4], so $X_{l+n, 3l+3n-3} = 0$ for δ -grading reasons, and e_4 has nowhere to live. The cancellation is responsible for the next (-1) in the sequence $d^{(n)}$, and e_2 and e_3 are responsible for the (2) following it.

On the other hand, if $n > l$, then X has two exceptional pairs in $t = l + n$ but $V_{l+n} = 0$, so none of the e_i can cancel. If $n = l$, the same argument holds: X must have three exceptional pairs in $t = l + n$ but $\dim V_{l+n}$ is only two, so none of the e_i can cancel. Thus e_1 and (say) e_2 add an exceptional pair to the lower summand, and e_3 and e_4 add an exceptional pair to the upper summand. These additions are precisely what we were expecting, completing the inductive argument. \square

5.2. $P(-l, l + 1, n)$ for even l . Now we will make a similar computation for $P(-l, l + 1, n)$ which we will use as the base case in the induction to follow. As with odd l , we do not need to start with $n = 0$. We can use $n = l$ instead; since $P(-l, l + 1, l) = P(-l, l, l + 1)$, the previous section tells us the formula for $Kh(P(-l, l + 1, l))$.

Lemma 5.2. *For $n \geq l + 1$, $Kh(P(-l, l + 1, n))$ is given by the formula in Theorem 2.9.*

Proof. Resolve the top crossing on the rightmost strand, a positive crossing. Our diagram for $P(-l, l + 1, n)$ has no negative crossings, while the unoriented resolution (a diagram for the unknot) has $l + n - 1$ negative crossings. The oriented resolution is a diagram for $P(-l, l + 1, n - 1)$. Hence the skein sequence is

$$\xrightarrow{f} q^{3l+3n-1} t^{l+n} Kh(U) \longrightarrow Kh(P(-l, l + 1, n)) \longrightarrow qKh(P(-l, l + 1, n - 1)) \xrightarrow{f},$$

where U is the unknot, with Khovanov polynomial $q + q^{-1}$. Denote the left, middle, and right terms by W , X , and V respectively. W has an exceptional pair in $t = l + n$ (in fact, W consists of this exceptional pair). If n is even, V has one exceptional pair in $t = 0$, and X has two exceptional pairs ($t = 0$ and $t = l + n$) since the linking number of the two components of $P(-l, l + 1, n)$ is $\frac{l+n}{2}$. If n is odd, V has two exceptional pairs ($t = 0$ and $t = l + n - 1$), and X has one exceptional pair in $t = 0$. It is clear that the upper and lower summands of V are based at the correct points to become the upper and lower summands we want for X , after some cancellations.

Figure 13 depicts some generators of V and W . For $t \geq l + n - 2$, V has only the generators shown. One can check this statement either inductively, if $n > l + 1$, or by looking at the formula for $P(-l, l + 1) = P(-l, l + 1, l)$ if $n = l + 1$.

If n is even then there is no cancellation since $V_{l+n,*} = 0$, $\dim W_{l+n,*} = 2$, and X has an exceptional pair in $t = l + n$. So $X = V \oplus W$, which agrees with our formulas.

If n is odd, pick a basis $\{e_1, e_2\}$ for W , in q -gradings $3l + 3n$ and $3l + 3n - 2$ respectively. Now, $V_{l+n-1, 3l+3n} = 0$; this follows from our formulas but can be seen most easily in Figure 13. Hence e_1 cannot cancel. On the other hand, e_2 must cancel: otherwise it would be in a knight's move, but $V_{l+n-1, 3l+3n-6} = 0$ and $V_{l+n+1, 3l+3n+2} = 0$. Thus our formula is verified. \square

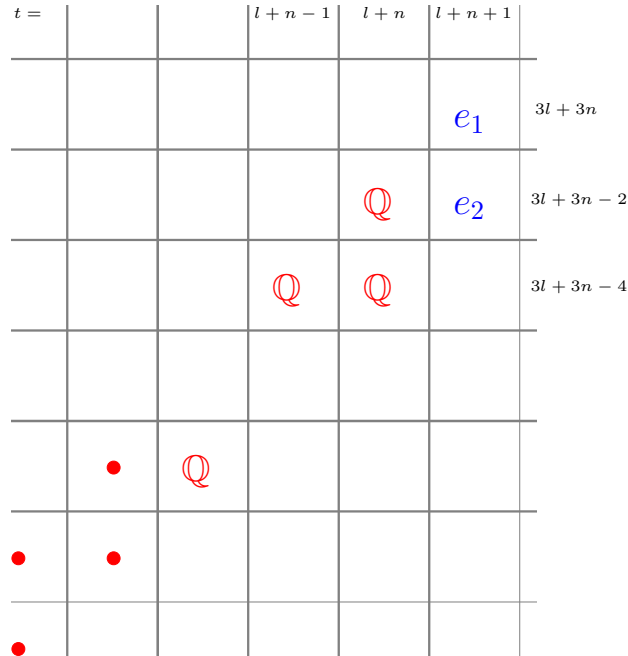


FIGURE 13. The case of odd n in Lemma 5.2. In $t \geq l + n - 2$, V is zero except for the copies of \mathbb{Q} shown. Red denotes V and blue denotes W .

5.3. $P(-l, m, n)$ for even l .

Theorem 5.3. *When l is even, $Kh(P(-l, m, n)_{RL})$ is given by the formula in Theorem 2.9.*

Proof. We already did the case of $m = l$, so we can induct on $m \geq l + 1$. The base case $m = l + 1$ was also done above. Assume that our formula holds for $P(-l, m - 1, n)$; we will prove it for $P(-l, m, n)$ using the skein exact sequence for the top crossing in the middle strand (a positive crossing). The diagram for $P(-l, m, n)$ has no negative crossings. The unoriented resolution is a diagram for the right-handed torus link $T_{n-l,2}$. If n is odd, the unoriented resolution diagram has $l + m - 1$ negative crossings, and $T_{n-l,2}$ is a knot. If n is even, $T_{n-l,2}$ is a 2-component link; orient it positively, so that the unoriented resolution diagram has $l + m - 1$ negative crossings. The sequence is

$$\xrightarrow{f} q^{3l+3m-1} t^{l+m} Kh(T_{n-l,2}) \longrightarrow Kh(P(-l, m, n)) \longrightarrow qKh(P(-l, m - 1, n)) \xrightarrow{f},$$

where $T_{n-l,2}$ is positively oriented when n is even. Call the three terms W , X , and V as usual, and write $V = L \oplus U$. As before, L and U are based at the correct points to become the upper and lower summands in our desired formula for X , after cancellations.

In fact, if m is even, there are no cancellations. To show this fact, first suppose $m > l + 2$. Nothing in W can cancel with anything in L for t -grading reasons, so $X = L \oplus X'$ where X' is a cancellation of $U \oplus W$.

Now, U has no exceptional pairs if n is odd, while if n is even, U has an exceptional pair in $l + n$. If n is odd, W has one exceptional pair in $t = l + m$, and if n is even it has an additional one in $t = m + n$. In either case, however, X' must have an exceptional pair everywhere that U or W does. Hence $E_{X'} \geq E_U + E_W$, so by Lemma 3.6, $X' = U \oplus W$. We are now done, since adding W to the upper summand U of V produces the upper summand we want for X .

The remaining case for even m is $m = l + 2$. The same argument applies as soon as we can show that nothing in W cancels with anything in L . But any such cancellation would need to occur between $t = 2l + 1$ and $t = 2l + 2$. In the column $t = 2l + 1$, L has a single generator, which is part of a knight's move. It cannot cancel because, if it did, its knight's move partner could not be part of any knight's move or exceptional pair in X . So we are done with even m .

If m is odd, a few cancellations occur. We may again consider cancellations X' of $U \oplus W$. If n is odd, W has one exceptional pair in its lowest t -grading, $t = l + m$. V has an exceptional pair in $t = l + m - 1$ and X does not, so by Lemma 3.8, there must be a standard cancellation between t -gradings $l + m - 1$ and $l + m$.

Now, any further cancellations would be cancellations of $U' \oplus W'$ as in Remark 3.7, but U' and W' have no exceptional pairs and are contained in $\delta = n + m - 1$ and $\delta = n + m - 3$, so Lemma 3.6 precludes any cancellations.

If n is even (but m is still odd), the same analysis applies to the lower exceptional pair of W . Now, however, W has an additional exceptional pair in $t = (n - l) + (l + m) =$

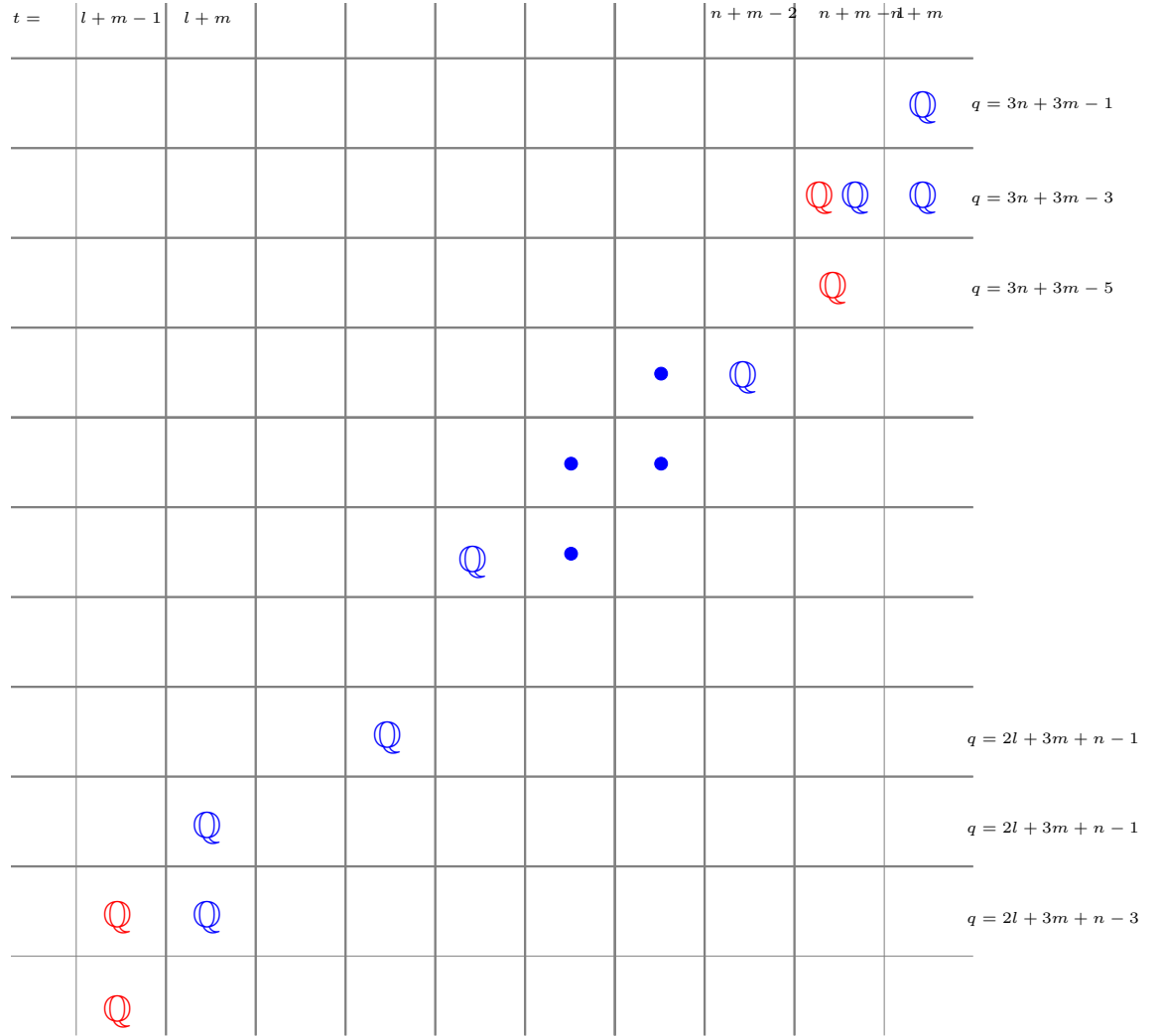


FIGURE 14. The case of odd n in Theorem 5.3. Red represents V and blue represents W .

$n+m$. Also, X has no exceptional pair in this t -grading, and V has an exceptional pair in $t = n+m-1$. Hence a standard cancellation must occur between $t = n+m-1$ and $t = n+m$ by Lemma 3.9.

As before, any further cancellations would be cancellations of $U' \oplus W'$ as in Remark 3.7, but U' and W' have no exceptional pairs and are contained in $\delta = n+m-1$ and $\delta = n+m-3$, so Lemma 3.6 precludes any cancellations. We can now easily check that we have completed the proof of our general formula. \square

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